

A Lower Bound for the Perron Root of a Nonnegative Matrix

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ABSTRACT

It is well known that the Perron root $r(A)$ of a nonnegative matrix A lies between the smallest and the largest row sum of A . We obtain a new lower bound for $r(A)$ by using a result of Kuharenko concerning a spectral property of a zero-trace matrix.

Before stating our result we repeat, for the convenience of the reader, Kuharenko's result, which we will need in our considerations.

THEOREM (Kuharenko [5]). *Let $A = (a_{ij})$ be an $n \times n$ real matrix, and let*

$$\text{Tr}(A) = \sum_{i=1}^n a_{ii} = 0 \quad (1)$$

and

$$A_2 = \sum_{1 \leq i < j \leq n} (a_{ii}a_{jj} - a_{ij}a_{ji}) > 0. \quad (2)$$

Then there exists at least one pair of complex-conjugate eigenvalues $\lambda_k = c_k \pm id_k$ of A such that

$$d_k \geq \sqrt{c_k^2 + \frac{2}{n}A_2}. \quad (3)$$

Proof. Taking into account the well-known formula

$$\mathrm{Tr}(A^2) = \mathrm{Tr}^2(A) - 2A_2, \quad (4)$$

(1), and (2), the existence of at least one pair $\lambda_k = c_k \pm id_k$ of complex-conjugate eigenvalues of A is obvious.

To prove the inequality (3) assume that there exist p real and q pairs of complex-conjugate eigenvalues of A , denoted by α_i [$i = 1(1)p$] and $\lambda_k = c_k \pm id_k$ [$k = 1(1)q$], respectively. Utilizing (4) and (1), a slight manipulation leads to

$$2 \sum_{k=1}^q (d_k^2 - c_k^2) = 2A_2 + \sum_{i=1}^p \alpha_i^2. \quad (5)$$

Observing that

$$2A_2 + \sum_{i=1}^p \alpha_i^2 = \sum_{k=1}^q \frac{1}{q} \left(2A_2 + \sum_{i=1}^p \alpha_i^2 \right), \quad (6)$$

we can write (5) in the equivalent form

$$\sum_{k=1}^q \left[d_k^2 - c_k^2 - \frac{1}{2q} \left(2A_2 + \sum_{i=1}^p \alpha_i^2 \right) \right] = 0. \quad (7)$$

From (7) it follows that for some k

$$d_k^2 - c_k^2 - \frac{1}{2q} \left(2A_2 + \sum_{i=1}^p \alpha_i^2 \right) \geq 0.$$

Therefore, noticing that $\alpha_i^2 \geq 0$ and $2q = n - p$, we obtain

$$d_k^2 \geq c_k^2 + \frac{1}{2q} \left(2A_2 + \sum_{i=1}^p \alpha_i^2 \right) \geq c_k^2 + \frac{1}{n-p} \cdot 2A_2 \geq c_k^2 + \frac{2}{n} A_2,$$

which completes the proof. ■

Now we proceed to our result.

THEOREM 1. *Let $A = (a_{ij})$ be an $n \times n$ nonnegative real matrix. Then for the Perron root $r(A)$ of A , the following inequality holds:*

$$r(A) \geq \min_i \{a_{ii}\} + \begin{cases} \max \left\{ \sqrt{\frac{2}{n-1} \sum_{1 \leq i < j \leq n} a_{ij} a_{ji}}, \right. \\ \left. \sqrt{\frac{1}{n} \operatorname{Tr} \left(\left(A - \min_i \{a_{ii}\} \cdot I \right)^2 \right)} \right\} & \text{for } n = 3, 5, 7, \dots, \\ \sqrt{\frac{1}{n} \operatorname{Tr} \left(\left(A - \min_i \{a_{ii}\} \cdot I \right)^2 \right)} & \text{for } n = 2, 4, 6, \dots, \end{cases} \quad (8)$$

where I is the identity matrix.

Proof. Starting with A , we define the matrix $C = (c_{ij})$ by

$$c_{ij} = \begin{cases} b & \text{for } i = j, \\ a_{ij} & \text{for } i < j, \\ -a_{ij} & \text{for } i > j \end{cases}$$

where $b = \min_i \{a_{ii}\}$. Then it is obvious that the following inequality holds:

$$|C - bI| \leq A - bI, \quad (9)$$

where the absolute value of $C - bI$ is taken componentwise. The above inequality implies [7, p. 57] that

$$\rho(C - bI) \leq r(A - bI), \quad (10)$$

where $\rho(C - bI)$ denotes the spectral radius of $C - bI$. Since $\operatorname{Tr}(C - bI) = 0$ and the sum of all the principal minors of $C - bI$ of order 2 is positive, we can apply the Kuharenko result to the matrix $C - bI$. Thus we obtain

$$\rho(C - bI) \geq \sqrt{\frac{2}{n} \sum_{1 \leq i < j \leq n} a_{ij} a_{ji}}.$$

Taking into account (10) and the proof of Kuharenko's result, and observing that in the case when n is odd $C - bI$ possesses at least one real eigenvalue (i.e. $p \geq 1$), we get

$$\begin{aligned} r(A) &= b + r(A - bI) \geq b + \rho(C - bI) \\ &\geq b + \begin{cases} \sqrt{\frac{2}{n-1} \sum_{1 \leq i < j \leq n} a_{ij} a_{ji}} & \text{for } n = 3, 5, 7, \dots, \\ \sqrt{\frac{2}{n} \sum_{1 \leq i < j \leq n} a_{ij} a_{ji}} & \text{for } n = 2, 4, 6, \dots \end{cases} \quad (11) \end{aligned}$$

On the other hand, denoting the eigenvalues of $A - bI$ by $\lambda_i(A - bI)$ [$i = 1(1)n$], we have

$$nr^2(A - bI) \geq \sum_{i=1}^n \lambda_i^2(A - bI) = \text{Tr}((A - bI)^2),$$

so that

$$r(A) = b + r(A - bI) \geq b + \sqrt{\frac{1}{n} \text{Tr}((A - bI)^2)}. \quad (12)$$

Combining (11) with (12) and noticing that

$$\frac{1}{n} \text{Tr}((A - bI)^2) \geq \frac{2}{n} \sum_{1 \leq i < j \leq n} a_{ij} a_{ji},$$

we get (8), proving the theorem (recall that $b = \min_i \{a_{ii}\}$). ■

REMARK 1. The bound is invariant under diagonal similarity transformations of A .

EXAMPLE. Let

$$A = \begin{bmatrix} 2 & 10 & 1 \\ 7 & 1 & 11 \\ 2 & 11 & 1 \end{bmatrix}.$$

In this case Theorem 1 yields

$$\begin{aligned} r(A) &\geq 1 + \max \left\{ \sqrt{\sum_{1 \leq i < j \leq 3} a_{ij} a_{ji}}, \sqrt{\frac{1}{3} \text{Tr}((A - I)^2)} \right\} \\ &= 1 + \max \left\{ \sqrt{193}, \sqrt{\frac{387}{3}} \right\} = 1 + \sqrt{193}. \end{aligned}$$

This result is better than that obtained by methods from [2], [4], [6], and [8] (the classical inequality of Frobenius yields $r(A) \geq 13$). This result is also better than that obtained by methods from [1] (for the partitions $M_1 = \{1\}$, $M_2 = \{2, 3\}$ and $M_1 = \{1, 2\}$, $M_2 = \{3\}$) and from [3] (for the partitions $\Pi_1 = \{\{1\}, \{2, 3\}\}$ and $\Pi_2 = \{\{1, 2\}, \{3\}\}$).

REMARK 2. It should be noted that the bound given by Theorem 1 is not always better than the one we could get from the methods that are known. For example, considering the matrix [6]

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 5 \end{bmatrix},$$

our Theorem 1 gives

$$r(A) \geq 1 + \sqrt{\frac{46}{3}},$$

while by the classical inequality of Frobenius and by the method from [3] (for the partition $\Pi = \{\{1, 2\}, \{3\}\}$), we get

$$r(A) \geq 5 \quad \text{and} \quad r(A) \geq 7, \quad \text{respectively.}$$

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